

Congruence of multilinear forms

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Abstract

Let

$$F: U \times \cdots \times U \rightarrow \mathbb{K}, \quad G: V \times \cdots \times V \rightarrow \mathbb{K}$$

be two n -linear forms with $n \geq 2$ on vector spaces U and V over a field \mathbb{K} . We say that F and G are symmetrically equivalent if there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ such that

$$F(u_1, \dots, u_n) = G(\varphi_{i_1} u_1, \dots, \varphi_{i_n} u_n)$$

for all $u_1, \dots, u_n \in U$ and each reordering i_1, \dots, i_n of $1, \dots, n$. The forms are said to be congruent if $\varphi_1 = \cdots = \varphi_n$.

Let F and G be symmetrically equivalent. We prove that

- (i) if $\mathbb{K} = \mathbb{C}$, then F and G are congruent;
- (ii) if $\mathbb{K} = \mathbb{R}$, $F = F_1 \oplus \cdots \oplus F_s \oplus 0$, $G = G_1 \oplus \cdots \oplus G_r \oplus 0$, and all summands F_i and G_j are nonzero and direct-sum-indecomposable, then $s = r$ and, after a suitable reindexing, F_i is congruent to $\pm G_i$.

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1 Introduction

Two matrices A and B over a field \mathbb{K} are called *congruent* if $A = S^T B S$ for some nonsingular S . Two matrix pairs (A_1, B_1) and (A_2, B_2) are called *equivalent* if $A_1 = R A_2 S$ and $B_1 = R B_2 S$ for some nonsingular R and S . Clearly, if A and B are congruent, then (A, A^T) and (B, B^T) are equivalent. Quite unexpectedly, the inverse statement holds for complex matrices too: if (A, A^T) and (B, B^T) are equivalent, then A and B are congruent [4, Chapter VI, §3, Theorem 3]. This statement was extended in [5, 6] to arbitrary systems of linear mappings and bilinear forms. In this article, we extend it to multilinear forms.

A *multilinear form* (or, more precisely, *n-linear form*, $n \geq 2$) on a finite dimensional vector space U over a field \mathbb{K} is a mapping $F: U \times \cdots \times U \rightarrow \mathbb{K}$ such that

$$\begin{aligned} F(u_1, \dots, u_{i-1}, au'_i + bu''_i, u_{i+1}, \dots, u_n) \\ = aF(u_1, \dots, u'_i, \dots, u_n) + bF(u_1, \dots, u''_i, \dots, u_n) \end{aligned}$$

for all $i \in \{1, \dots, n\}$, $a, b \in \mathbb{K}$, and $u_1, \dots, u'_i, u''_i, \dots, u_n \in U$.

Definition 1. Let

$$F: U \times \cdots \times U \rightarrow \mathbb{K}, \quad G: V \times \cdots \times V \rightarrow \mathbb{K} \quad (1)$$

be two n -linear forms.

(a) F and G are called *equivalent* if there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ such that

$$F(u_1, \dots, u_n) = G(\varphi_1 u_1, \dots, \varphi_n u_n)$$

for all $u_1, \dots, u_n \in U$.

(b) F and G are called *symmetrically equivalent* if there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ such that

$$F(u_1, \dots, u_n) = G(\varphi_{i_1} u_1, \dots, \varphi_{i_n} u_n) \quad (2)$$

for all $u_1, \dots, u_n \in U$ and each reordering i_1, \dots, i_n of $1, \dots, n$.

(c) F and G are called *congruent* if there exists a linear bijection $\varphi: U \rightarrow V$ such that

$$F(u_1, \dots, u_n) = G(\varphi u_1, \dots, \varphi u_n).$$

for all $u_1, \dots, u_n \in U$.

The *direct sum* of forms (1) is the multilinear form

$$F \oplus G: (U \oplus V) \times \dots \times (U \oplus V) \rightarrow \mathbb{K}$$

defined as follows:

$$(F \oplus G)(u_1 + v_1, \dots, u_n + v_n) := F(u_1, \dots, u_n) + G(v_1, \dots, v_n)$$

for all $u_1, \dots, u_n \in U$ and $v_1, \dots, v_n \in V$.

We will use the internal definition: if $F: U \times \dots \times U \rightarrow \mathbb{K}$ is a multilinear form, then $F = F_1 \oplus F_2$ means that there is a decomposition $U = U_1 \oplus U_2$ such that

- (i) $F(x_1, \dots, x_n) = 0$ as soon as $x_i \in U_1$ and $x_j \in U_2$ for some i and j ,
- (ii) $F_1 = F|_{U_1}$ and $F_2 = F|_{U_2}$ are the restrictions of F to U_1 and U_2 .

A multilinear form $F: U \times \dots \times U \rightarrow \mathbb{K}$ is *indecomposable* if for each decomposition $F = F_1 \oplus F_2$ and the corresponding decomposition $U = U_1 \oplus U_2$ we have $U_1 = 0$ or $U_2 = 0$.

Our main result is the following theorem.

Theorem 2. (a) *If two multilinear forms over \mathbb{C} are symmetrically equivalent, then they are congruent.*

(b) *If two multilinear forms F and G over \mathbb{R} are symmetrically equivalent and*

$$F = F_1 \oplus \dots \oplus F_s \oplus 0, \quad G = G_1 \oplus \dots \oplus G_r \oplus 0$$

are their decompositions such that all summands F_i and G_j are nonzero and indecomposable, then $s = r$ and, after a suitable reindexing, each F_i is congruent to G_i or $-G_i$.

The statement (a) of this theorem is proved in the next section. We prove (b) in the end of Section 3 basing on Corollary 11, in which we argue that every n -linear form $F: U \times \dots \times U \rightarrow \mathbb{K}$ with $n \geq 3$ over an arbitrary field \mathbb{K} decomposes into a direct sum of indecomposable forms uniquely up to congruence of summands. Moreover, if $F = F_1 \oplus \dots \oplus F_s \oplus 0$ is a decomposition

in which F_1, \dots, F_s are nonzero and indecomposable, and $U = U_1 \oplus \dots \oplus U_s \oplus U_0$ is the corresponding decomposition of U , then the sequence of subspaces $U_1 + U_0, \dots, U_s + U_0, U_0$ is determined by F uniquely up to permutations of $U_1 + U_0, \dots, U_s + U_0$.

2 Symmetric equivalence and congruence

In this section, we prove Theorem 2(a) and the following theorem, which is a weakened form of Theorem 2(b).

Theorem 3. *If two multilinear forms F and G over \mathbb{R} are symmetrically equivalent, then there are decompositions*

$$F = F_1 \oplus F_2, \quad G = G_1 \oplus G_2$$

such that F_1 is congruent to G_1 and F_2 is congruent to $-G_2$.

Its proof is based on two lemmas.

Lemma 4. (a) *Let T be a nonsingular complex matrix having a single eigenvalue. Then*

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{C}[x]: \quad f(T)^m = T^{-1}.$$

(b) *Let T be a real matrix whose set of eigenvalues consists of one positive real number or a pair of distinct conjugate complex numbers. Then*

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{R}[x]: \quad f(T)^m = T^{-1}. \quad (3)$$

Proof. (a) Let T be a nonsingular complex matrix with a single eigenvalue λ . Since the matrix $T - \lambda I$ is nilpotent (this follows from its Jordan canonical form), the substitution of T for x into the Taylor expansion

$$\begin{aligned} x^{-\frac{1}{m}} &= \lambda^{-\frac{1}{m}} + \left(-\frac{1}{m}\right) \lambda^{-\frac{1}{m}-1}(x - \lambda) \\ &\quad + \frac{1}{2!} \left(-\frac{1}{m}\right) \left(-\frac{1}{m} - 1\right) \lambda^{-\frac{1}{m}-2}(x - \lambda)^2 + \dots \end{aligned} \quad (4)$$

gives some matrix

$$f(T), \quad f(x) \in \mathbb{C}[x], \quad (5)$$

satisfying $f(T)^m = T^{-1}$.

(b) Let T be a square real matrix. If it has a single eigenvalue that is a positive real number λ , then all coefficients in (4) are real, so the matrix (5) satisfies (3).

Let T have only two eigenvalues

$$\lambda = a + ib, \quad \bar{\lambda} = a - ib \quad (a, b \in \mathbb{R}, b > 0). \quad (6)$$

It suffices to prove (3) for any matrix that is similar to T over \mathbb{R} , so we may suppose that T is the real Jordan matrix

$$T = R^{-1} \begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix} R = \begin{bmatrix} aI + F & bI \\ -bI & aI + F \end{bmatrix}, \quad R := \begin{bmatrix} I & -iI \\ I & iI \end{bmatrix},$$

in which $J = \lambda I + F$ is a direct sum of Jordan blocks with the same eigenvalue λ (and so F is a nilpotent upper triangular matrix).

It suffices to prove that

$$\forall m \in \mathbb{N} \quad \exists f(x) \in \mathbb{R}[x]: \quad f(J)^m = J^{-1} \quad (7)$$

since such $f(x)$ satisfies (3):

$$\begin{aligned} f(T)^m &= f(R^{-1}(J \oplus \bar{J})R)^m = R^{-1}f(J \oplus \bar{J})^m R \\ &= R^{-1}(f(J)^m \oplus \overline{f(J)^m})R = R^{-1}(J \oplus \bar{J})^{-1}R = T^{-1}. \end{aligned}$$

The matrix F is nilpotent, so the substitution of $J = \lambda I + F$ into the Taylor expansion (4) gives some matrix $g(J)$ with $g(x) \in \mathbb{C}[x]$ satisfying $g(J)^m = J^{-1}$. Represent $g(x)$ in the form

$$g(x) = g_0(x) + ig_1(x), \quad g_0(x), g_1(x) \in \mathbb{R}[x].$$

It suffices to prove that J reduces to iI by a finite sequence of polynomial substitutions

$$J \longmapsto h(J), \quad h(x) \in \mathbb{R}[x].$$

Indeed, their composite is some polynomial $p(x) \in \mathbb{R}[x]$ such that $p(J) = iI$, and then $f(x) := g_0(x) + p(x)g_1(x) \in \mathbb{R}[x]$ satisfies (7):

$$f(J)^m = (g_0(J) + p(J)g_1(J))^m = (g_0(J) + ig_1(J))^m = g(J)^m = J^{-1}.$$

First, we replace J by $b^{-1}(J - aI)$ (see (6)) making $J = iI + F$. Next, we replace J by

$$\frac{3}{2}J + \frac{1}{2}J^3 = \frac{3}{2}(iI + F) + \frac{1}{2}(-iI - 3F + 3iF^2 + F^3) = iI + F',$$

where $F' := (3iF^2 + F^3)/2$. The degree of nilpotency of F' is less than the degree of nilpotency of F ; we repeat the last substitution until obtain iI . \square

Definition 5. Let $G: V \times \cdots \times V \rightarrow \mathbb{K}$ be an n -linear form. We say that a linear mapping $\tau: V \rightarrow V$ is G -selfadjoint if

$$G(v_1 \dots, v_{i-1}, \tau v_i, v_{i+1} \dots, v_n) = G(v_1 \dots, v_{j-1}, \tau v_j, v_{j+1} \dots, v_n)$$

for all $v_1, \dots, v_n \in V$ and all i and j .

If τ is G -selfadjoint, then for every $f(x) \in \mathbb{K}[x]$ the linear mapping $f(\tau)$ is G -selfadjoint too.

Lemma 6. Let $G: V \times \cdots \times V \rightarrow \mathbb{K}$ be a multilinear form over a field \mathbb{K} and let $\tau: V \rightarrow V$ be a G -selfadjoint linear mapping. If

$$V = V_1 \oplus \cdots \oplus V_s \tag{8}$$

is a decomposition of V into a direct sum of τ -invariant subspaces such that the restrictions $\tau|_{V_i}$ and $\tau|_{V_j}$ of τ to V_i and V_j have no common eigenvalues for all $i \neq j$, then

$$G = G_1 \oplus \cdots \oplus G_s, \quad G_i := G|_{V_i}. \tag{9}$$

Proof. It suffices to consider the case $s = 2$. To simplify the formulas, we assume that G is a bilinear form. Choose $v_1 \in V_1$ and $v_2 \in V_2$, we must prove that $G(v_1, v_2) = G(v_2, v_1) = 0$.

Let $f(x)$ be the minimal polynomial of $\tau|_{V_2}$. Since $\tau|_{V_1}$ and $\tau|_{V_2}$ have no common eigenvalues, $f(\tau|_{V_1}): V_1 \rightarrow V_1$ is a bijection, so there exists $v'_1 \in V_1$ such that $v_1 = f(\tau)v'_1$. Since τ is G -selfadjoint, $f(\tau)$ is G -selfadjoint too, and so

$$\begin{aligned} G(v_1, v_2) &= G(f(\tau)v'_1, v_2) = G(v'_1, f(\tau)v_2) \\ &= G(v'_1, f(\tau|_{V_2})v_2) = G(v'_1, 0v_2) = G(v'_1, 0) = 0. \end{aligned}$$

Analogously, $G(v_2, v_1) = 0$. \square

Proof of Theorem 2(a). Let n -linear forms (1) over $\mathbb{K} = \mathbb{C}$ be symmetrically equivalent; this means that there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ satisfying (2) for each reordering i_1, \dots, i_n of $1, \dots, n$. Let us prove by induction that F and G are congruent. Assume that $\varphi := \varphi_1 = \dots = \varphi_t$ for some $t < n$ and prove that there exist linear bijections

$$\psi_1 = \dots = \psi_t = \psi_{t+1}, \psi_{t+2}, \dots, \psi_n: U \rightarrow V$$

such that

$$F(u_1, \dots, u_n) = G(\psi_{i_1} u_1, \dots, \psi_{i_n} u_n) \quad (10)$$

for all $u_1, \dots, u_n \in U$ and each reordering i_1, \dots, i_n of $1, \dots, n$.

By (2) and since $\varphi_1, \dots, \varphi_n$ are bijections, for every pair of distinct indices i, j and for all $u_i, u_j \in U$ and $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n \in V$, we have

$$\begin{aligned} G(v_1, \dots, v_{i-1}, \varphi u_i, v_{i+1}, \dots, v_{j-1}, \varphi_{t+1} u_j, v_{j+1}, \dots, v_n) \\ = G(v_1, \dots, v_{i-1}, \varphi_{t+1} u_i, v_{i+1}, \dots, v_{j-1}, \varphi u_j, v_{j+1}, \dots, v_n). \end{aligned} \quad (11)$$

Denote $v_i := \varphi_{t+1} u_i$ and $v_j := \varphi_{t+1} u_j$. Then (11) takes the form

$$G(\dots, \varphi \varphi_{t+1}^{-1} v_i, \dots, v_j, \dots) = G(\dots, v_i, \dots, \varphi \varphi_{t+1}^{-1} v_j, \dots);$$

this means that the linear mapping $\tau := \varphi \varphi_{t+1}^{-1}: V \rightarrow V$ is G -selfadjoint.

Let $\lambda_1, \dots, \lambda_s$ be all distinct eigenvalues of τ and let (8) be the decomposition of V into the direct sum of τ -invariant subspaces such that every $\tau_i := \tau|_{V_i}$ has a single eigenvalue λ_i . Lemma 6 ensures (9). For every $f_i(x) \in \mathbb{C}[x]$, the linear mapping $f_i(\tau_i): V_i \rightarrow V_i$ is G_i -selfadjoint. Using Lemma 4(a), we take $f_i(x)$ such that $f_i(\tau_i)^{t+1} = \tau_i^{-1}$. Then

$$\rho := f_1(\tau_1) \oplus \dots \oplus f_s(\tau_s): V \rightarrow V$$

is G -selfadjoint and $\rho^{t+1} = \tau^{-1}$.

Define

$$\psi_1 = \dots = \psi_{t+1} := \rho \varphi, \quad \psi_{t+2} := \varphi_{t+2}, \dots, \psi_n := \varphi_n. \quad (12)$$

Since ρ is G -selfadjoint and

$$\rho^{t+1} \varphi = \tau^{-1} \varphi = (\varphi \varphi_{t+1}^{-1})^{-1} \varphi = \varphi_{t+1},$$

we have

$$\begin{aligned}
G(\psi_1 u_1, \dots, \psi_n u_n) &= G(\rho \varphi u_1, \dots, \rho \varphi u_t, \rho \varphi u_{t+1}, \varphi_{t+2} u_{t+2}, \dots, \varphi_n u_n) \\
&= G(\varphi u_1, \dots, \varphi u_t, \rho^{t+1} \varphi u_{t+1}, \varphi_{t+2} u_{t+2}, \dots, \varphi_n u_n) \\
&= G(\varphi_1 u_1, \dots, \varphi_n u_n) = F(u_1, \dots, u_n).
\end{aligned}$$

So (10) holds for $i_1 = 1, i_2 = 2, \dots, i_n = n$. The equality (10) for an arbitrary reordering i_1, \dots, i_n of $1, \dots, n$ is proved analogously. \square

Proof of Theorem 3. Let n -linear forms (1) over $\mathbb{K} = \mathbb{R}$ be symmetrically equivalent; this means that there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ satisfying (2) for each reordering i_1, \dots, i_n of $1, \dots, n$. Assume that $\varphi := \varphi_1 = \dots = \varphi_t$ for some $t < n$. Just as in the proof of Theorem 2(a), $\tau := \varphi \varphi_{t+1}^{-1}$ is G -selfadjoint. Let (8) be the decomposition of V into the direct sum of τ -invariant subspaces such that every $\tau_p := \tau|_{V_p}$ has a single real eigenvalue λ_p or a pair of conjugate complex eigenvalues

$$\lambda_p = a_p + ib_p, \quad \bar{\lambda}_p = a_p - ib_p, \quad b_p > 0,$$

and $\lambda_p \neq \lambda_q$ if $p \neq q$. Lemma 6 ensures the decomposition (9).

Define the G -selfadjoint linear bijection

$$\varepsilon = \varepsilon_1 1_{V_1} \oplus \dots \oplus \varepsilon_s 1_{V_s}: V \rightarrow V,$$

in which $\varepsilon_i = -1$ if λ_i is a negative real number, and $\varepsilon_i = 1$ otherwise. Replacing φ_{t+1} by $\varepsilon \varphi_{t+1}$, we obtain τ without negative real eigenvalues. But the right-hand member of the equality (2) may change its sign on some subspaces V_p . To preserve (2), we also replace φ_{t+2} with $\varepsilon \varphi_{t+2}$ if $t+1 < n$ and replace $G = G_1 \oplus \dots \oplus G_s$ (see (9)) with

$$\varepsilon_1 G_1 \oplus \dots \oplus \varepsilon_s G_s \tag{13}$$

if $t+1 = n$. By Lemma 4(b), for every i there exists $f_i(x) \in \mathbb{R}[x]$ such that $f_i(\tau_i)^{t+1} = \tau_i^{-1}$. Define

$$\rho = f_1(\tau_1) \oplus \dots \oplus f_s(\tau_s): V \rightarrow V,$$

then $\rho^{t+1} = \tau^{-1}$. Reasoning as in the proof of Theorem 2(a), we find that (10) with (13) instead of G holds for the linear mappings (12). \square

We say that two systems of n -linear forms

$$F_1, \dots, F_s: U \times \cdots \times U \rightarrow \mathbb{K}, \quad G_1, \dots, G_s: V \times \cdots \times V \rightarrow \mathbb{K}$$

are *equivalent* if there exist linear bijections $\varphi_1, \dots, \varphi_n: U \rightarrow V$ such that

$$F_i(u_1, \dots, u_n) = G_i(\varphi_1 u_1, \dots, \varphi_n u_n).$$

for each i and for all $u_1, \dots, u_n \in U$. These systems are said to be *congruent* if $\varphi_1 = \cdots = \varphi_n$.

For every n -linear form F , we construct the system of n -linear forms

$$\mathcal{S}(F) = \{F^\sigma \mid \sigma \in S_n\}, \quad F^\sigma(u_1, \dots, u_n) := F(u_{\sigma(1)}, \dots, u_{\sigma(n)}), \quad (14)$$

where S_n denotes the set of all substitutions on $1, \dots, n$.

The next corollary is another form of Theorem 2(a).

Corollary 7. *Two multilinear forms F and G over \mathbb{C} are congruent if and only if the systems of multilinear forms $\mathcal{S}(F)$ and $\mathcal{S}(G)$ are equivalent.*

To each substitution $\sigma \in S_n$, we assign some $\varepsilon(\sigma) \in \{1, -1\}$. Generalizing the notions of symmetric and skew-symmetric bilinear forms, we say that an n -linear form F is ε -*symmetric* if $F^\sigma = \varepsilon(\sigma)F$ for all $\sigma \in S_n$. If G is another ε -symmetric n -linear form, then $\mathcal{S}(F)$ and $\mathcal{S}(G)$ are equivalent if and only if F and G are equivalent. So the next corollary follows from Corollary 7.

Corollary 8. *Two ε -symmetric multilinear forms over \mathbb{C} are equivalent if and only if they are congruent.*

3 Direct decompositions

Every bilinear form over \mathbb{C} or \mathbb{R} decomposes into a direct sum of indecomposable forms uniquely up to congruence of summands; see the classification of bilinear forms in [1, 2, 3, 6]. In [6, Theorem 2 and §2] this statement was extended to all systems of linear mappings and bilinear forms over \mathbb{C} or \mathbb{R} . The next theorem shows that a stronger statement holds for n -linear forms with $n \geq 3$ over all fields.

Theorem 9. *Let $F: U \times \cdots \times U \rightarrow \mathbb{K}$ be an n -linear form with $n \geq 3$ over a field \mathbb{K} .*

(a) Let $F = F' \oplus 0$ and let F' have no zero direct summands. If $U = U' \oplus U_0$ is the corresponding decomposition of U , then U_0 is uniquely determined by F and F' is determined up to congruence.

(b) Let F have no zero direct summands and let $F = F_1 \oplus \cdots \oplus F_s$ be its decomposition into a direct sum of indecomposable forms. If $U = U_1 \oplus \cdots \oplus U_s$ is the corresponding decomposition of U , then the sequence U_1, \dots, U_s is determined by F uniquely up to permutations.

Proof. (a) The subspace U_0 is uniquely determined by F since U_0 is the set of all $u \in U$ satisfying

$$F(u, x_1, \dots, x_{n-1}) = F(x_1, u, x_2, \dots, x_{n-1}) = \cdots = F(x_1, \dots, x_{n-1}, u) = 0$$

for all $x_1, \dots, x_{n-1} \in U$.

Let $F = F' \oplus 0 = G' \oplus 0$ be two decompositions in which F' and G' have no zero direct summands, and let $U = U' \oplus U_0 = V' \oplus U_0$ be the corresponding decompositions of U . Choose bases u_1, \dots, u_m of U' and v_1, \dots, v_m of V' such that $u_1 - v_1, \dots, u_m - v_m$ belong to U_0 . Then

$$F(u_{i_1}, \dots, u_{i_n}) = F(v_{i_1}, \dots, v_{i_n})$$

for all $i_1, \dots, i_n \in \{1, \dots, m\}$, and so the linear bijection

$$\varphi: U' \longrightarrow V', \quad u_1 \mapsto v_1, \dots, u_m \mapsto v_m,$$

gives the congruence of F' and G' .

(b) Let $F: U \times \cdots \times U \rightarrow \mathbb{K}$ be an n -linear form with $n \geq 3$ that has no zero direct summands, let

$$F = F_1 \oplus \cdots \oplus F_s = G_1 \oplus \cdots \oplus G_r \tag{15}$$

be two decompositions of F into direct sums of indecomposable forms, and let

$$U = U_1 \oplus \cdots \oplus U_s = V_1 \oplus \cdots \oplus V_r \tag{16}$$

be the corresponding decompositions of U .

Put

$$d_1 = \dim U_1, \dots, d_s = \dim U_s \tag{17}$$

and choose two bases

$$u_1, \dots, u_m \in U_1 \cup \cdots \cup U_s, \quad v_1, \dots, v_m \in V_1 \cup \cdots \cup V_r \tag{18}$$

of the space U with the following ordering of the first basis:

$$u_1, \dots, u_{d_1} \text{ is a basis of } U_1, \quad u_{d_1+1}, \dots, u_{d_1+d_2} \text{ is a basis of } U_2, \quad \dots \quad (19)$$

Let C be the transition matrix from u_1, \dots, u_m to v_1, \dots, v_m . Partition it into s horizontal and s vertical strips of sizes d_1, d_2, \dots, d_s . Since C is nonsingular, by interchanging its columns (i.e., reindexing v_1, \dots, v_m) we make nonsingular all diagonal blocks. Changing the bases (19), we make elementary transformations within the horizontal strips of C and reduce it to the form

$$C = \begin{bmatrix} I_{d_1} & C_{12} & \dots & C_{1s} \\ C_{21} & I_{d_2} & \dots & C_{2s} \\ \dots & \dots & \dots & \dots \\ C_{s1} & C_{s2} & \dots & I_{d_s} \end{bmatrix}. \quad (20)$$

It suffices to prove that $u_1 = v_1, \dots, u_m = v_m$, that is,

$$C_{pq} = 0 \quad \text{if } p \neq q. \quad (21)$$

Indeed, by (18) $v_1 \in V_p$ for some p . Since F_1 is indecomposable, if $d_1 > 1$ then $u_1, u_2 \in U_1$ and so

$$F(\dots, u_1, \dots, u_2, \dots) \neq 0 \quad \text{or} \quad F(\dots, u_2, \dots, u_1, \dots) \neq 0 \quad (22)$$

for some elements of U denoted by points. If (21) holds, then $u_1 = v_1$ and $u_2 = v_2$. Since $v_1 \in V_p$, (22) ensures that $v_2 \notin V_q$ for all $q \neq p$, and so $v_2 \in V_p$. This means that $U_1 \subset V_p$. Therefore, after a suitable reindexing of V_1, \dots, V_s we obtain $U_1 \subset V_1, \dots, U_r \subset V_r$. By (16), $r = s$ and $U_1 = V_1, \dots, U_r = V_r$; so the statement (b) follows from (22).

Let us prove (21). For each substitution $\sigma \in S_n$, the n -linear form F^σ defined in (14) can be given by the n -dimensional matrix

$$\mathbb{A}^\sigma = [a_{ij\dots k}^\sigma]_{i,j,\dots,k=1}^m, \quad a_{ij\dots k}^\sigma := F^\sigma(u_i, u_j, \dots, u_k),$$

in the basis u_1, \dots, u_m , or by the n -dimensional matrix

$$\mathbb{B}^\sigma = [b_{ij\dots k}^\sigma]_{i,j,\dots,k=1}^m, \quad b_{ij\dots k}^\sigma := F^\sigma(v_i, v_j, \dots, v_k),$$

in the basis v_1, \dots, v_m . Then for all $x_1, \dots, x_n \in U$ and their coordinate vectors $[x_i] = (x_{1i}, \dots, x_{mi})^T$ in the basis u_1, \dots, u_m , we have

$$F^\sigma(x_1, \dots, x_n) = \sum_{i,j,\dots,k=1}^m a_{ij\dots k}^\sigma x_{i1} x_{j2} \dots x_{kn}. \quad (23)$$

If $C = [c_{ij}]$ is the transition matrix (20), then

$$b_{i'j' \dots k'}^\sigma = \sum_{i,j, \dots, k=1}^m a_{ij \dots k}^\sigma c_{ii'} c_{jj'} \cdots c_{kk'}. \quad (24)$$

By (15), $a_{ij\dots k}^\sigma = F^\sigma(u_i, u_j, \dots, u_k) \neq 0$ only if all u_i, u_j, \dots, u_k belong to the same space U_l . Hence \mathbb{A}^σ and, analogously, \mathbb{B}^σ decompose into the direct sums of n -dimensional matrices:

$$\mathbb{A}^\sigma = \mathbb{A}_1^\sigma \oplus \cdots \oplus \mathbb{A}_s^\sigma, \quad \mathbb{B}^\sigma = \mathbb{B}_1^\sigma \oplus \cdots \oplus \mathbb{B}_r^\sigma, \quad (25)$$

in which every \mathbb{A}_i^σ has size $d_i \times \cdots \times d_i$ and every \mathbb{B}_j^σ has size $\dim V_j \times \cdots \times \dim V_j$.

We prove (21) using induction in n .

Base of induction: $n = 3$. The 3-dimensional matrices \mathbb{A}^σ and \mathbb{B}^σ can be given by the sequences of m -by- m matrices

$$\begin{aligned} A_1^\sigma &= [a_{ij1}^\sigma]_{i,j=1}^m, \dots, A_m^\sigma = [a_{ijm}^\sigma]_{i,j=1}^m, \\ B_1^\sigma &= [b_{ij1}^\sigma]_{i,j=1}^m, \dots, B_m^\sigma = [b_{ijm}^\sigma]_{i,j=1}^m; \end{aligned}$$

we call these matrices the *layers* of \mathbb{A}^σ and \mathbb{B}^σ . The equality (23) takes the form

$$F^\sigma(x_1, x_2, x_3) = [x_1]^T (A_1^\sigma x_{13} + \cdots + A_m^\sigma x_{m3}) [x_2] \quad (26)$$

for all $x_1, x_2, x_3 \in U$ and their coordinate vectors $[x_i] = (x_{1i}, \dots, x_{mi})^T$ in the basis u_1, \dots, u_m . Put

[illegible]

By (24),

$$b_{i'j'k'}^\sigma = \sum_{i,j=1}^m (a_{ij1}^\sigma c_{1k'} + \cdots + a_{ijm}^\sigma c_{mk'}) c_{ii'} c_{jj'},$$

and so

$$B_1^\sigma = C^T H_1^\sigma C, \dots, B_m^\sigma = C^T H_m^\sigma C. \quad (28)$$

Partition $\{1, \dots, m\}$ into the subsets

$$\mathcal{I}_1 = \{1, \dots, d_1\}, \quad \mathcal{I}_2 = \{d_1 + 1, \dots, d_1 + d_2\}, \quad \dots \quad (29)$$

(see (17)). By (25), if $k \in \mathcal{I}_q$ for some q , then the k^{th} layer of \mathbb{A}^σ has the form

$$A_k^\sigma = 0_{d_1} \oplus \cdots \oplus 0_{d_{q-1}} \oplus \tilde{A}_k^\sigma \oplus 0_{d_{q+1}} \oplus \cdots \oplus 0_{d_s}, \quad (30)$$

in which \tilde{A}_k^σ is d_q -by- d_q . So by (27) and since all diagonal blocks of the matrix (20) are the identity matrices,

$$H_k^\sigma = \sum_{i \in \mathcal{I}_1} \tilde{A}_i^\sigma c_{ik} \oplus \cdots \oplus \sum_{i \in \mathcal{I}_{q-1}} \tilde{A}_i^\sigma c_{ik} \oplus \tilde{A}_k^\sigma \oplus \sum_{i \in \mathcal{I}_{q+1}} \tilde{A}_i^\sigma c_{ik} \oplus \cdots \oplus \sum_{i \in \mathcal{I}_s} \tilde{A}_i^\sigma c_{ik}. \quad (31)$$

We may suppose that

$$\sum_{\sigma \in S_3} \sum_{i=1}^m \text{rank } A_i^\sigma \geq \sum_{\sigma \in S_3} \sum_{i=1}^m \text{rank } B_i^\sigma; \quad (32)$$

otherwise we interchange the direct sums in (15). By (30) and (28),

$$\sum_{\sigma \in S_3} \sum_{i=1}^m \text{rank } \tilde{A}_i^\sigma \geq \sum_{\sigma \in S_3} \sum_{i=1}^m \text{rank } H_i^\sigma; \quad (33)$$

Let us fix distinct p and q and prove that $C_{pq} = 0$ in (20). Due to (31), (33), and (30),

$$\forall k \in \mathcal{I}_q : \quad \sum_{i \in \mathcal{I}_p} A_i^\sigma c_{ik} = 0. \quad (34)$$

Replacing in this sum each A_i^σ by the basis vector u_i , we define

$$u := \sum_{i \in \mathcal{I}_p} u_i c_{ik} \in U_p. \quad (35)$$

Since

$$[u] = (0, \dots, 0, c_{d+1,k}, \dots, c_{d+d_p,k}, 0, \dots, 0)^T, \quad d := d_1 + \cdots + d_{p-1},$$

by (26) and (34) we have $F^\sigma(x, y, u) = 0$ for all $x, y \in U_p$. This equality holds for all substitutions $\sigma \in S_3$, hence

$$F(u, x, y) = F(x, u, y) = F(x, y, u) = 0, \quad (36)$$

and so $F|u\mathbb{K}$ is a zero direct summand of $F_p = F|U_p$. Since F_p is indecomposable, $u = 0$; that is, $c_{d+1,k} = \cdots = c_{d+d_p,k} = 0$. These equalities hold for all $k \in \mathcal{I}_q$, hence $C_{pq} = 0$. This proves (21) for $n = 3$.

Induction step. Let $n \geq 4$ and assume that (21) holds for all $(n-1)$ -linear forms.

The n -dimensional matrices \mathbb{A}^σ and \mathbb{B}^σ can be given by the sequences of $(n-1)$ -dimensional matrices

$$\begin{aligned} A_1^\sigma &= [a_{i_{\dots j 1} \dots}^\sigma]_{i_{\dots j} = 1}^m, \dots, A_m^\sigma = [a_{i_{\dots j m} \dots}^\sigma]_{i_{\dots j} = 1}^m, \\ B_1^\sigma &= [b_{i_{\dots j 1} \dots}^\sigma]_{i_{\dots j} = 1}^m, \dots, B_m^\sigma = [b_{i_{\dots j m} \dots}^\sigma]_{i_{\dots j} = 1}^m. \end{aligned}$$

By (24),

$$\begin{aligned} b_{i' \dots j' 1}^\sigma &= \sum_{i, \dots, j} (a_{i \dots j 1}^{\sigma} c_{11} + \cdots + a_{i \dots j m}^{\sigma} c_{m1}) c_{ii'} \cdots c_{jj'} \\ &\vdots \\ b_{i' \dots j' m}^\sigma &= \sum_{i, \dots, j} (a_{i \dots j 1}^{\sigma} c_{1m} + \cdots + a_{i \dots j m}^{\sigma} c_{mm}) c_{ii'} \cdots c_{jj'} \end{aligned} \quad (37)$$

Due to (25) and analogous to (30), each A_k^σ with $k \in \mathcal{I}_q$ (see (29)) is a direct sum of $d_1 \times \cdots \times d_1, \dots, d_s \times \cdots \times d_s$ matrices, and only the q^{th} summand \tilde{A}_k^σ may be nonzero. This implies (31) for each k and for H_k^σ defined in (27).

For each $(n-1)$ -linear form G , denote by $s(G)$ the number of *nonzero* summands in a decomposition of G into a direct sum of indecomposable forms; this number is uniquely determined by G due to induction hypothesis. Put $s(M) := s(G)$ if G is given by an $(n-1)$ -dimensional matrix M . By (37), the set of $(n-1)$ -linear forms given by $(n-1)$ -dimensional matrices (27) is congruent to the set of $(n-1)$ -linear forms given by $B_1^\sigma, \dots, B_m^\sigma$. Hence

$$s(H_1^\sigma) = s(B_1^\sigma) \ , \ \dots, \ s(H_m^\sigma) = s(B_m^\sigma). \quad (38)$$

We suppose that

$$\sum_{\sigma \in S_n} \sum_{k=1}^m s(A_k^\sigma) \geq \sum_{\sigma \in S_n} \sum_{k=1}^m s(B_k^\sigma),$$

otherwise we interchange the direct sums in (15). Then by (38)

$$\sum_{\sigma \in S_n} \sum_{k=1}^m s(\tilde{A}_k^\sigma) \geq \sum_{\sigma \in S_n} \sum_{k=1}^m s(H_k^\sigma). \quad (39)$$

Let us fix distinct p and q and prove that $C_{pq} = 0$ in (20). By (31),

$$s(H_k^\sigma) = s(\tilde{A}_k^\sigma) + \sum_{p \neq q} s\left(\sum_{i \in \mathcal{I}_p} \tilde{A}_i^\sigma c_{ik}\right)$$

for each $k \in \mathcal{I}_q$. Combining it with (39), we have

$$\sum_{i \in \mathcal{I}_p} A_i^\sigma c_{ik} = \sum_{i \in \mathcal{I}_p} \tilde{A}_i^\sigma c_{ik} = 0$$

for each $k \in \mathcal{I}_q$. Define u by (35). As in (36), we obtain

$$F(u, x, \dots, y) = F(x, u, \dots, y) = \dots = F(x, \dots, y, u) = 0$$

for all $x, \dots, y \in U_p$ and so $F|_{u\mathbb{K}}$ is a zero direct summand of $F_p = F|_{U_p}$. Since F_p is indecomposable, $u = 0$; so $C_{pq} = 0$. This proves (21) for $n > 3$. \square

Remark 10. Theorem 9(b) does not hold for bilinear forms: for example, the matrix of scalar product is the identity in each orthonormal basis of a Euclidean space. This distinction between bilinear and n -linear forms with $n \geq 3$ may be explained by the fact that decomposable bilinear forms are more frequent. Let us consider forms in a two-dimensional vector space. To decompose a bilinear form, we must make zero two entries in its 2×2 matrix. To decompose a trilinear form, we must make zero six entries in its $2 \times 2 \times 2$ matrix. In both the cases, these zeros are made by transition matrices, which have four entries.

Corollary 11. *Let $F: U \times \dots \times U \rightarrow \mathbb{K}$ be an n -linear form with $n \geq 3$ over a field \mathbb{K} . If*

$$F = F_1 \oplus \dots \oplus F_s \oplus 0 \tag{40}$$

and the summands F_1, \dots, F_s are nonzero and indecomposable, then these summands are determined by F uniquely up to congruence. Moreover, if $U = U_1 \oplus \dots \oplus U_s \oplus U_0$ is the corresponding decomposition of U , then the sequence of subspaces

$$U_1 + U_0, \dots, U_s + U_0, U_0 \tag{41}$$

is determined by F uniquely up to permutations of $U_1 + U_0, \dots, U_s + U_0$.

Proof of Theorem 2(b). For $n = 2$ this theorem was proved in [6, Section 2.1] (and was extended to arbitrary systems of forms and linear mappings in [6, Theorem 2]). For $n \geq 3$ this theorem follows from Theorem 3 and Corollary 11. \square

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